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PRIMITIVE IDEMPOTENTS IN SEMIGROUPS

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To the memory of Professor Otto Stešnfeld

Abstract. In this paper we study certain general properties of primitive and 0-primitive idempotents, and the obtained results we apply to 0-inversive, 0-B-inversive, E-inversive and B-inversive semigroups. We also determine some conditions under which a left ideal of a semigroup with zero a generated by a nonzero idempotent is left 0-simple. The obtained results generalize some results by Venkatesan, Steinfeld, Bogdanović and Milić, Bogdanović and Ćirić, Mitsch and Petrich and others.

1. Introduction and preliminaries

An idempotent a of a semigroup S is called *primitive* if it is minimal in the set $E(S)$ of all idempotents of S with respect to the natural paxtial order on idempotents, i.e. if for every $f \in E(S)$, $f = ef = fe$ implies $f = e$. But, theory of primitive idempotents for a semigroup with zero is trivial, and from this reason we define the concept of primitivity only for semigroups without zero. If S is a semigroup with zero 0, then we define a nonzero idempotent a of S to be 0-*primitive* if it is minimal in the set $E^*(S)$ of all nonzero idempotents of S, i.e. if $e \in E^*(S)$ and for every $f \in E^*(S)$, by $f = ef = fe$ it follows $f = e$.

The concepts of primitivity and 0-primitivity play an outstanding role in theory of semigroups. For example, they are used to define completely simple and completely 0-simple semigroups. Semigroups without zero all of whose idempotents are primitive, called *primitive semigroups*, and semigroups with zero all of whose nonzero idempotents are 0-primitive, called 0-*primitave semigroups*, have been a subject of interest of many authors. Primitive regular semigroups are exactly completely simple ones, whereas 0-primitive regular semigroups were characterized as orthogonal sums of completely 0-simple semigroups, by Venkatesan in [21] and Steinfeld in [17].

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These semigroups have been also studied by Hall in [9], Lallement and Petrich in [14], Preston in [16], and more information about it can be found in the books by Clifford and Preston [8], Steinfeld [18], Bogdanović and Ćirić [5] and Howie [12]. Primitive π-regular semigroups were characterized by Bogdanović and Milić in [3] as nil-extensions of completely simple semigroups, and 0-primitive π-regular semigroups were described by Bogdanović and Ćirić in [4] as certain extensions of 0-primitive regular semigroups.

The main our aim is to give certain general properties of primitive and 0-primitive idempotents and to describe primitive and 0-primitive semigroups in some classes more general than the class of π-regular semigroups, such as E-inversive, B-inversive, 0-inversive and O-B-inversive semigroups. On the other hand, it is known that the primitivity and 0-primitivity of idempotents are closely related to the minimality and 0-minimality of twosided, onesided and bi-ideals of a semigroup, and we study this relationship, too. For example, we determine some necessary and sufficient conditions for a left ideal of a semigroup with zero generated by a nonzero idempotent to be a left 0-simple semigroup. The obtained results generalize the above mentioned results of Venkatesan, Steinfeld, Bogdanović and Milić, Bogdanović and Ciric, Lallement and Petrich [14] and others, as well as the results of Mitsch and Petrich, announced in [15], concerning 0-primitive 0-inversive and primitive E-inversive semigroups.

Throughout the paper, $Reg(S)$ and $E(S)$ denote the set of all regular elements and the set of all idempotents of a semigroup S, respectively, whereas $S = S^0$ means that S is a semigroup with zero 0. A semigroup S is called *unipotent* if it has a single idempotent, wheras a semigroup $S = S^0$ with a single nonzero idempotent is called 0-*unipotent.* A subsemigroup K of a semigroup S is called *full* if $E(S) \subseteq K$. Let a semigroup $S = S^0$ be given. Then $E^*(S) = E(S)\backslash\{0\}$, and if K is a subset of S, then $A_S(K) = \{a \in S | aK = Ka = \{0\}\}$, $A^l_S(K) = \{a \in S | aK = \{0\}\}$ and $A^r_S(K) = \{a \in S | Ka = \{0\}\}$. If $S\backslash\{0\}$ is a group (left group, union of groups), then S is said to be a *group (left group, union of groups) with zero.*

For undefined notions and notations we refer to the books by Clifford and Preston [7] and [8[], Steinfeld [18], Bogdanović [1], Howie [12] and Bogdanović and Ćirić [5].

In the further work we need the following known results.

Lemma 1. [18] *Let a be a nonzero idempotent of a semigroup $S = S^0$. Then Se is a 0-minimal left ideal of S if and only if a is 0-primitive and $Se \subseteq Reg(S)$.*

Lemma 2. [7] *If L is a 0-minimal left ideal of a semigroup $S = S^\circ$ such that $L^2 \neq \{0\}$, then $L = Sa$, for any $a \in L\backslash\{0\}$.*

Lemma 3. [7] *If $S = S^0$ is a left 0-simple semigroup, then $S \backslash \{0\}$ is a left simple subsemigroup of S.*

Lemma 4. [2] *Let S be a semigroup without zero. Then the following conditions are equivalent:*

(i) *there exists $e \in E(S)$ such that eSe is a group;*

(ii) *there exists $e \in E(S)$ such that Se is a left group;*

(iii) *there exists $e \in E(S)$ such that eS is a right group;*

(iii) *S has a completely simple kernel.*

2. The results

The first theorem of this paper characterizes 0-primitive idempotents of a semigroup with zero.

Theorem 1. *Let e be a nonzero idempotent of a semigroup $S = S^0$. Then the following conditions are equivalent:*

(i) *e is 0-primitive;*

(ii) *eSe is 0-unipotent;*

(iii) $(\forall f \in E^*(Se))ef = e$;

(iv) *$E^*(Se)$ is a left zero band;*

(v) $(\forall f \in E^*(eS))fe = e$;

(vi) *$E^*(eS)$ is a right zero band.*

Proof. (i)⇒(ii). Let e be a 0-primitive idempotent of S and let $f \in E^*(eSe)$. Then $0 \neq f = ef = fe$, whence $f = e$. Thus eSe is 0-unipotent.

(ii)⇒(i). Let $f \in E^*(S)$ such that $f = ef = fe$. Then $f \in eSe$ and since eSe is 0-unipotent, then we have that $f = e$. Hence, e is 0-primitive.

(ii)⇒(iii). Let $f \in E^*(Se)$. Then $f = fe$, whence $ef = efe \in eSe$. By this and the hypothesis it follows that $(ef)^2 = ef \in \{0, e\}$. If $ef = 0$, then $f = f^2 = (fe)f = f(ef) = 0$, which is not possible. Therefore, $ef = e$.

(iii)⇒(i). Let $f \in E^*(S)$ such that $f = ef = fe$. Then $f \in E^*(Se)$, whence $ef = e$, i.e. $f = e$. Therefore, we have proved that e is 0-primitive.

(iii)⇒(iv). Let f, $g \in E^*(Se)$. Then $f = fe$ and by the hypothesis we have that $fg = (fe)g = f(eg) = fe = f$.

(iv)⇒(iii). This implication is evident.

The remaining equivalences can be proved in a similar manner. □

As a consequence of the previous theorem we obtain the following result concerning primitive idempotents of a semigroup without zero.

Corollary 1. *Let e be an idempotent of a semigroup S without zero. Then the following conditions are equivalent:*

(i) *e is primitive;*

(ii) *eSe is a unipotent monoid;*

(iii) *$E(Se)$ is a left zero band;*

(iv) *$E(eS)$ is a right zero band.*

A semigroup S is called *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax = (ax)^2$. These semigroups were introduced by Thierrin in [19] and they have been considered by a number of authors as a generalization of regular, π-regular[1)] and many other important kinds of semigroups. But, any semigroup with zero is E-inversive, so in this case is interesting to consider another related concept. Namely, a semigroup $S = S^0$ is called *0-inversive* if for every $a \in S\backslash\{0\}$ there exists $x \in S$ such that $ax = (ax)^2 \neq 0$. This concept was first studied by Lallement in [13] and Lallement and Petrich in [14], and since by many other authors. Let us note that the definitions of E-inversive and 0-inversive semigroups are self-dual. In other words, a semigroup S is E-inversive if and only if for every $a \in S$ there exists $y \in S$ such that $ya = (ya)^2$, or equivalently, if for every $a \in S$ there exists $z \in S$ such that $z = zaz$ (see [6]). Similarly, a semigroup $S = S^0$ is 0-inversive if for every $a \in S\backslash\{0\}$ there exists $y \in S$ such that $ya = (ya)^2 \neq 0$, or equivalently, if for every $a \in S\backslash\{0\}$ there exists $z \in S$ such that $z = zaz \neq 0$.

We characterize 0-inversive semigroups by the following theorem.

Theorem 2. *The following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is 0-inversive;*

(ii) *every nonzero ideal of S contains a nonzero idempotent;*

(iii) *every nonzero deft (right) ideal of S contains a nonzero idempotent.*

Proof. (i)⇒(ii) and (i)⇒(iii). Let S be 0-inversive and let $a \in S\backslash\{0\}$. Then there exists $x \in S$ such that $ax \in R(a) \subseteq J(a)$ and $ax = (ax)^2 \neq 0$. Thus every nonzero right and every nonzero twosided ideal of S have a nonzero idempotent. Similarly we prove that every nonzero left ideal of S has a nonzero idempotent.

(iii)⇒(ii). This implication is evident.

(ii)⇒(i). Let $a \in S\backslash\{0\}$. Then $\mathrm{J}(a) \neq \{0\}$ and there exists an idempotent $e \neq 0$ such that $e \in J(a)$. This means that there exist $x, y \in S$ such that $e = xay \neq 0$, whence $xay = (xay)^2 \neq 0$. Set $u = yxayx$. We have that

$$0 \neq xay = (xay)^3 = xa(yxayx)ay = xauay,$$

whence $u \neq 0$. Since

$$uau = (yxayx)a(yxayx) = yxayx = u \neq 0,$$

we then have that $au \in E^*(S)$. Therefore, S is 0-inversive. □

[1)]Note that a semigroup S is called π-regular if for any element of S, some its power is regular.

Note that the conditions (ii) and (iii) of the previous theorem can be replaced by the corresponding conditions concerning principal left, right and twosided ideals.

As we noted before, the 0-primitivity of an idempotent is closely related to the 0-minimality of left and right ideals generated by it. Namely, the following is true.

Lemma 5. *If a is a nonzero idempotent of a semigroup* $S = S^0$ *such that the left ideal* Se(right ideal eS) *of* S *generated by a as 0-minimal, then e is a 0-primitive idempotent.*

Proof. For a proof see Lemma 6.38 of [8].

The converse of the previous lemma is not true, as the next example shows.

Example 1. Let S be a semigroup given by the following presentation:

$$S{=}<a, e, 0 | a^2{=}0, \quad e^2{=}e, \quad ae{=}0, \quad ea{=}a, \quad Oa{=}a0{=}e0{=}Oe{=}0^2{=}0> .$$

Then e is a 0-primitive idempotent of S, but $eS = S$, so eS is not a 0-minimal right ideal of S.

This example motivates us to give the following definition. A nonzero idempotent e of a semigroup $S = S^0$ which generates a 0-minimal left (right) ideal is called *left (right) completely 0-primitive*, and e is *completely 0-primitive* if it is both left and right completely 0-primitive. A semigroup $S = S^0$ is called *(left, right) completely 0-primitive* if all of its nonzero idempotents are (left, right) *completely 0-primitive.* By the following theorem we describe 0-primitive idempotents of 0-inversive semigroups.

Theorem 3. *Let* $S = S^0$ *be a 0-inversive semigroup and let* $e \in E^*(S)$. *Then the following conditions are equivalent:*

(i) *e is 0-primitive;*

(ii) *Se is a 0-minimal left ideal;*

(iii) *eS is a 0-minimal right ideal;*

(iv) *eSe is a group with zero.*

Proof. We shall prove only the equivalence (i) $\Longleftrightarrow$ (ii). Let e be a 0-primitive idempotent and let $a \in Se$, $a \neq 0$. Then $a = ae$. On the other hand, there exists $y \in S$ such that $y \neq 0$ and $ya = (ya)^2 \neq 0$. It is clear that $ya \in Sa = Sae \subseteq Se$. Since $ya \neq 0$, we have that $ya \in E^*(Se)$. By (iii) of Theorem 1 we have that $eya = e$. Now $a = ae = a(eya) = a(ey)a$, so $a \in Reg(S)$. Thus, $Se \subseteq Reg(S)$ and by Lemma 1 we have that Se is a 0-minimal left ideal of S.

The converse is an immediate consequence of Lemma 5. □

Next we characterize 0-inversive semigroups having 0-primitive idempotents.

Theorem 4. *Let $S = S^0$ be a 0-inversive semigroup. Then the following conditions are equivalent:*

(i) *S has a 0-primitive idempotent;*

(ii) *S has a 0-primitive regular left ideal;*

(iii) *S has a 0-primitive regular right ideal.*

Proof. We shall prove only the equivalence (i) $\Longleftrightarrow$ (ii).

Let e be a 0-primitive idempotent of S. Then by Theorem 3, Se is a 0-minimal Ieft ideal of S. Let P be the set of all 0-primitive idempotents of S and let $L = \cup_{e \in P}(Se)$. Since $Se = See \subseteq Le \subseteq Se$, we have that $L = \cup_{e \in P} Le$. Now by Theorem 6.39 of [8] it follows that L is a 0-primitive regular Left ideal of S.

The converse is evident. □

By the previous two theorems we obtain the following one.

Theorem 5. *Let $S = S^0$ be a 0-inversive semigroup. Then the following conditions are equivalent:*

(i) *S is 0-primitive;*

(ii) *eSe is a group with zero, for any $e \in E^*(S)$;*

(iii) *S has a full 0-primitive regular ideal;*

(iv) *S has a full 0-primitive regular left (right) ideal;*

(v) *S is completely 0-prirraitive.*

Proof. (i)$\Rightarrow$(iii). By Theorem 3 it follows that Se is a 0-minimal left ideal of S and eS is a 0-minimal right ideal of S, for every $e \in E^*(S)$. On the other hand, according to Lemma 1 we have that

$$L = \bigcup_{e \in E^*(S)} Se \subseteq Reg(S) \text{ and } R = \bigcup_{e \in E^*(S)} eS \subseteq Reg(S).$$

Since every $E^*(S)$ is 0-primitive, we then have that $Reg(S) \subseteq L \cap R$. Therefore, $Reg(S) = L = R$ is a 0-primitive regular ideal of S.

(i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (v). This follows by Theorem 3.

(i) $\Longleftrightarrow$ (iv). This is a consequence of Theorem 4.

(iii) $\Longleftrightarrow$ (i). This implication is clear. □

As a consequence of the previous theorem we obtain the structural characterization of 0-primitive 0-inversive semigroups announced by Mitsch and Petrich in [15].

Corollary 2. *A semigroup $S = S^0$ is a 0-primitive 0-inversive semigroup if and only if it is an ideal extension of a 0-primitive regular semigroup K by a semigroup without nonzero idempotents and $A_S(K) = \{0\}$.*

Note that the condition $A_S(K) = \{0\}$ can be replaced by the conditions $A^l_S(K) = \{0\}$ and $A^r_S(K) = \{0\}$.

We introduce a special kind of 0-inversive semigroups as follows. A semigroup $S = S^0$ is *O-B-inversive* if for every $a \in S\backslash\{0\}$ there exists $x \in S$ such that $axa = (axa)^2 \neq 0$. Clearly, every O-B-inversive semigroup is 0-inversive. It is easy to verify that the five element Brandt semigroup B_2 is 0-inversive and that this semigroup is not O-B-inversive.

The following theorem gives a characterization of O-B-inversive semigroups.

Theorem 6. *The following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is O-B-inversive;*

(ii) *every nonzero left (right) ideal of S is 0-inversive;*

(iii) *every nonzero bi-ideal of S contains a nonzero idempotent.*

Proof. (i)⇒(ii). Let L be a nonzero left ideal of S and let $a \in L\backslash\{0\}$. Then there exists $x \in S$ such that $axa = (axa)^2 \neq 0$, and hence $0 \neq axa \in Sa \subseteq L$. Therefore, by Theorem 2 we have that S is 0-inversive.

(ii)⇒(i). Let $a \in S\backslash\{0\}$. Then there exists $y \in L(a)$ such that $ay = (ay)^2 \neq 0$. Since $y = a$ or $y = sa$, for some $s \in S\backslash\{0\}$, we have that $aa = (aa)^2 \neq 0$, or $asa = (asa)^2 \neq 0$. Thus S is O-B-inversive.

(i)⇒(iii) . This is evident.

(iii)⇒(i). Let $a \in S\backslash\{0\}$. By the hypothesis, the principal bi-ideal $B(a) = \{a, a^2\} \cup aSa$ generated by a has a nonzero idempotent e, and we have that $e = a$ or $e = a^2$ or $e = axa$, for some $x \in S$. By this it follows that $0 \neq e = a^3 = a^6$ or $0 \neq e = a^4 = a^8$ or $0 \neq e = axa = (axa)^2$. Therefore, S is O-B-inversive. □

Note that the conditions (ii) and (iii) of the previous theorem can be replaced by the corresponding conditions concerning only the principal left, right and bi-ideals.

A very interesting property of O-B-inversive semigroups is presented by the following lemma.

Lemma 6. *If a semigroup $S = S^0$ is O-B-inversive, then $a^2 \neq 0$, for any $a \in S\backslash\{0\}$.*

Proof. Let $a \in S\backslash\{0\}$. If $a^2 = 0$, then $(axa)^2 = 0$ for each $x \in S$, which is not possible because S is O-B-inversive. Thus $a^2 \neq 0$ for any $a \in S\backslash\{0\}$. □

The previous lemma yields the following one.

Lemma 7. *The following conditions on a semigroup $S = S^0$ are equivalent:*

(i) *S is completely 0-simple and O-B-inversive;*

(ii) *S is completely 0-simple and $a^2 \neq 0$ for each $a \in S\backslash\{0\}$;*

(iii) *S is 0-simple and a union of groups;*

(iv) *S is a completely simple semigroup with zero.*

Proof. (i)⇒(ii). This follows by Lemma 6.
(ii)⇒(iii). This follows by Theorem 2.52 of [7].
(iii)⇒(iv) and (iv)⇒(i). These implications are obvious. □

Now we are able to characterize 0-primitive O-B-inversive semigroups.

Theorem 7. *Let $S = S^0$ be a O-B-inversive semigroup. Then S is 0-primitive if and only if S has a full 0-primitive ideal which is a union of groups with zero.*

Proof. This follows immediately by Theorem 5 and Lemma 7. □

By the previous theorems concerning 0-inversive semigroups we can deduce the corresponding results concerning E-inversive semigroups without zero. Namely, the following theorem, proved by Higgins in [11]is true.

Theorem 8. *Let S be a semigroup without zero. Then the following conditions are equivalent:*

(i) *S is E-inversive;*

(ii) *every ideal of S has an idempotent;*

(iii) *every left (right) ideal of S has an idempotent.*

A semigroup S without zero is B-inversive if for every $a \in S$ there exists $x \in S$ such that $axa = (axa)^2$, [1]. These semigroups can be characterized as follows.

Theorem 9. *Let S be a semigroup without zero. Then the following conditions are equivalent:*

(i) *S is B-inversive;*

(ii) *every left (right) ideal of S is an E-inversive subsemigroup of S;*

(iii) *every bi-ideal of S contains an idempotent.*

The proof of the above theorem is similar to the proof of Theorem 6 and it will be omitted.

The conditions of the previous two theorems can be replaced by the corresponding conditions concerning principal twosided, left, right and biideals.

Now we are ready to characterize primitive E-inversive and B-inversive semigroups. -

Theorem 10. *Let S be a semigroup without zero. There the fodlowirg corcditious are equivalent:*

(i) *S is E-inversive and primitive;*

(ii) *S is B-iuversive and primitive;*

(iii) *S is E-inversive and eSe is a group, for each $e \in E(S)$;*

(iv) *S has a full completely simple kernel.*

Proof. (i)⇒(iii). Let S be an E-inversive primitive semigroup, let $e \in E(S)$ and $a \in eSe$. Then $a = ea = ae$ and there exists $y \in S$such that

$ay = (ay)^2$ and $ya = (ya)^2$. Then $aey = (aey)^2$, whence $aey = aeyeaey$ and finally $aeye = (aeye)^2$. Similarly we obtain that $eyea = (eyea)^2$. By Theorem 2, eSe is unipotent and we have that $eyea = aeye = e$. Therefore, eSe is a group.

(iii)⇒(iv). This follows by Lemma 4.

(iv)⇒(ii). Let K be a full completely simple kernel of S, let $a \in S$ and $b \in K$. Then $aba \in K$ and $aba = abayaba$, $abay = yaba$, for some $y \in K$. By this it follows that $abay^2aba = (abay^2aba)^2$; so S is B-inversive. Since K is full, we have that S is primitive.

(ii)⇒(i): This implication is evident. □

As a consequence we obtain the following structural characterization of primitive E-inversive semigroups due to Mitsch and Petrich [15].

Corollary 3. *A semigroup S without zero is a primitive E-inversive semigroup if and only if it is are ideal extension of a completely simple semigroup K by a semigroup without nonzero idempotents.*

In the further text we consider an interesting problem which concerns 0-minimal left ideals of a semigroup. Namely, it is well-known that if an ideal I of a semigroup $S = S^0$ is 0-minimal, then either $I^2 = \{0\}$ or else I is a 0-simple semigroup. But, left ideals do not have such property, as was shown by the following example.

Example 2. Let S be a semigroup given by the following table:

	0	e	f	a	b
0	0	0	0	0	0
e	0	e	0	a	0
f	0	0	f	0	b
a	0	0	a	0	e
b	0	b	0	f	0

Then $Se = \{0, e, b\}$ is a 0-minimal left ideal and $(Se)^2 \neq \{0\}$, but it is not a left 0-simple semigroup, because $\{0, b\}$ is a left ideal of Se.

Our next goal is to determine certain conditions under which a left ideal of a semigroup $S = S^0$ generated by a nonzero idempotent a is left 0-simple. It is clear that such left ideal must be 0-minimal, which is by Lemma 1 equivalent to the condition that a is 0-primitive and $Se \subseteq Reg(S)$. The latest condition means that any element from Se is regular in S, but in the general case, it is not necessarily regular in Se. This confirms the element b from Example 2 which is regular in S but not in Se. We shall see that the regularity in Se, i.e. the condition $Se = Reg(Se)$ is crucial in the further work.

Theorem 11. *Let e be a nonzero idempotent of a semigroup $S = S^0$. Then the followang conditions are equivalent:*

(i) *Se is left 0-simple;*

(ii) *Se is a left group with zero;*
(iii) *e is 0-primitive and $Se = Reg(Se)$;*
(iv) *Se is a 0-minimal left ideal of S and $Se = Reg(Se)$;*
(v) *eSe is a group with zero and $Se = Reg(Se)$;*
(vi) *$E^*(Se)$ is a left zero and and $Se = Reg(Se)$;*
(vii) *eSe as 0-unipotent and $Se = Reg(Se)$.*

Proof. (i) $\iff$ (ii). The implication (i)$\Rightarrow$(ii) is evident, whereas the implication (ii)$\Rightarrow$(i) is an immediate consequence of Theorem 1.27 of [7].

(ii)$\Rightarrow$(iv). Let Se be a left group with zero. Then it is clear that Se is a 0-minimal left ideal of S. Since $Se\setminus\{0\}$ is a left group, we have that both $Se\setminus\{0\}$ and Se are regular semigroups, that is $Se = Reg(Se)$.

(iv)$\Rightarrow$(iii). This follows immediately by Lemma 5.

(iii)$\Rightarrow$(i). Let (iii) hold. Then it is clear that $(Se)^2 \neq \{0\}$, and by Lemma 1 it follows that Se is a 0-minimal left ideal of S. By Lemma 2 we have that $Se = Sa$, for every $a \in Se\setminus\{0\}$, whereas by $Se = Reg(Se)$ it follows that $a = ae$ and $a \in a(Se)a$, so $Se = Sa \subseteq S(aSea) \subseteq Sea \subseteq Seae \subseteq Se$, which means that $Se = Sea$, for every $a \in Se\setminus\{0\}$. Therefore, Se is a left 0-simple semigroup.

(iii)$\Rightarrow$(v). Let (iii) hold. By (iii) $\iff$ (i) and $eSe \subseteq Se$ it follows that $eSe\setminus\{0\}$ is a subsemigroup of $Se\setminus\{0\}$. Let $a \in eSe\setminus\{0\}$. Then $ae = ea = a$, and by $eSe \subseteq Se = Reg(Se)$ it follows that $a = axa$, for some $x \in S$. Now we obtain that $a(exe)a = (ae)x(ea) = axa = a$ and $exe \in eSe\setminus\{0\}$, so we conclude that $eSe\setminus\{0\}$ is a regular semigroup. Finally, by Theorem 1 it follows that $eSe\setminus\{0\}$ has only one idempotent, which means that $eSe\setminus\{0\}$ is a group, i.e. eSe is a group with zero.

(v)$\Rightarrow$(vii). This implication is obvious.

(iii) $\iff$ (vi) and (iii) $\iff$ (vii). These equivalence are immediate consequences of Theorem 1. □

The next theorem describes 0-minimal left ideals which are generated by nonzero idempotents.

Theorem 12. *The following conditions on a nonzero left ideal L of a seraigroup $S = S^0$ are equivalent:*
(i) *L is a 0-minimal left ideal of S and $L \subseteq Reg(S)$;*
(ii) *$E^*(L)$ is a left zero band and $L \subseteq Reg(S)$;*
(iii) *L is a 0-minimal left ideal of S generated by a nonzero idempotent.*

Proof. (i)$\Rightarrow$(iii). Let (i) hold and let $a \in L\setminus\{0\}$. By the hypothesis $L \subseteq Reg(S)$ it follows that there exists $x \in S$ such that $a = axa$ and $xa \in E^*(L)$. On the other hand, $Sxa \subseteq SL \subseteq L$, i.e. Sxa is a left ideal of S contained in L, and since L is a 0-minimal left ideal of S, then $Sxa = L$, which is to be proved.

(iii)$\Rightarrow$(ii). Let L be 0-minimal and let $L = Se$, for some $e \in E^*(L)$. By Lemma 1, e is 0-primitive and $L \subseteq Reg(S)$, and finally, by Theorem 1, $E^*(L)$ is a left zero band.

(ii)$\Rightarrow$(i). Let (ii) hold and let K be a nonzero left ideal of S contained in L. Then $K \subseteq Reg(S)$, so for any $a \in K\backslash\{0\}$ there exists $x \in S$ such that $a = axa$, and then $xa \in SK \subseteq K$, i.e. $xa \in E^*(K) \subseteq E^*(L)$. It is clear that $LK \subseteq SK \subseteq K$, so we have that K is a lefi ideal of L. Consider an arbitrary $b \in L\backslash\{0\}$. Then $b \in Reg(S)$, i.e. $b = byb$, for some $y \in S$. By this it follows that $yb \in E^*(L)$. Using the hypothesis that $E(L)$ is a left zero band we obtain that $yb = yb \cdot xa$, whence $b = byb = b \cdot yb \cdot xa \in Sxa \subseteq K$. Therefore, we have proved that $L = K$, so we conclude that L is a 0-minimal left ideal of S. □

Finally, as a consequence of two previous theorem we obtain the following result.

Theorem 13. *The following conditions on a left ideal L of a semigroup $S = S^0$ are equivalent:*

(i) *L is a 0-minimal left ideal of S and $L = Reg(L)$;*

(ii) *$E^*(L)$ is a left zero band and $L = Reg(L)$;*

(iii) *L is a left group with zero;*

(iv) *L is a left 0-simple and has a nonzero idempotent.*

Proof. (i) $\Longleftrightarrow$ (ii). This is an immediate consequence of Theorem 12.

(i)$\Rightarrow$(iii). Let L be 0-minimal and $L = Reg(L)$. By Theorem 12, $L = Se$ for some $e \in E^*(L)$, and by Theorem 11 we have that L is a left group with zero.

(iii)$\Rightarrow$(ii) and (iii)$\Rightarrow$(iv). These implications are evident.

(iv)$\Rightarrow$(iii). This follows by Lemma 3 and Theorem 1.27 of [7]. □

3. References

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